

LOCALLY ARC-TRANSITIVE GRAPHS OF VALENCE $\{3, 4\}$ WITH TRIVIAL EDGE KERNEL

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ABSTRACT. In this paper we consider connected locally G -arc-transitive graphs with vertices of valence 3 and 4, such that the kernel $G_{uv}^{[1]}$ of the action of an edge-stabiliser on the neighbourhood $\Gamma(u) \cup \Gamma(v)$ is trivial. We find nineteen finitely presented groups with the property that any such group G is a quotient of one of these groups. As an application, we enumerate all connected locally arc-transitive graphs of valence $\{3, 4\}$ on at most 350 vertices whose automorphism group contains a locally arc-transitive subgroup G with $G_{uv}^{[1]} = 1$.

1. INTRODUCTION

An arc in a simple graph Γ is an ordered pair of adjacent vertices of Γ . Let Γ be graph and G a group of automorphisms of Γ . Then Γ is said to be *G -arc-transitive* provided that G acts transitively on the set of arcs of Γ . Similarly, Γ is said to be *locally G -arc-transitive* if for every vertex v the stabiliser G_v of v acts transitively on the set of all arcs of Γ with the initial vertex being v . A graph Γ is arc-transitive if it is $\text{Aut}(\Gamma)$ -arc-transitive. In this paper, we shall be particularly interested in the structure of the vertex-stabilisers (and thus of the group G itself) in certain locally G -arc-transitive graphs. All the graphs in this paper are assumed to be connected.

If Γ is a connected locally G -arc-transitive graph, then it is well known that G is transitive on the edges of Γ and that it has at most two orbits on the vertex set $V(\Gamma)$. If G is transitive on $V(\Gamma)$, then it is in fact arc-transitive. On the other hand, if G has two orbits on $V(\Gamma)$, then we say that Γ is *genuinely* locally G -arc-transitive. In this case it can be shown that Γ is bipartite and that the two orbits form the bipartition of Γ . Furthermore, the group G is generated by a pair of stabilisers G_u and G_v of two adjacent vertices $u, v \in V(\Gamma)$.

If G is an automorphism group of a graph Γ and $v \in V(\Gamma)$, then we let $G_v^{\Gamma(v)}$ denote the permutation group induced by the action of G_v on the neighbourhood $\Gamma(v)$ of the vertex v , and let $G_v^{[1]}$ denote the kernel of this action (that is, $G_v^{[1]}$ is the group of all those elements of G that fix v and each of its neighbours). Similarly, for an edge uv of Γ let $\Gamma(uv) = \Gamma(u) \cup \Gamma(v) \setminus \{u, v\}$ and let $G_{uv}^{\Gamma(uv)}$ denote the permutation group induced by the action of G_{uv} on $\Gamma(uv)$. Observe that the kernel of this action is the intersection $G_u^{[1]} \cap G_v^{[1]}$, which will be denoted by $G_{uv}^{[1]}$ and called the *edge kernel* of the group G . Note that $G_v^{\Gamma(v)} \cong G_v/G_v^{[1]}$ and $G_{uv}^{\Gamma(uv)} \cong G_{uv}/G_{uv}^{[1]}$.

Since in a locally G -arc-transitive graph Γ the group G has at most two orbits on $V(\Gamma)$ it follows that the valence function can take only two values, say d_1 and d_2 ; the graph is then said to be *biregular* of valence $\{d_1, d_2\}$.

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If one of the two valences, say d_2 , is 2, then it is easy to see that the graph can be obtained from a G -arc-transitive graph of valence d_1 by subdividing each edge of that graph (where that graph is allowed to have parallel edges in the case when the original graph contains cycles of length 4). In this sense, the case of genuinely locally G -arc-transitive graphs of valence $\{2, d\}$ is equivalent to the family of G -arc-transitive graphs of valence d and has, as such, received much attention in the past; in particular, the structure of the vertex-stabiliser G_v has been determined for the cases $d = 3$ (see [8, 13, 27]) and $d = 4$ and 5 (see [12, 21, 28]).

The case where $d_1 = d_2 = 3$ was studied in several papers, most notably by Goldschmidt in [16], where it was proved that the group G must be a quotient of one of the 15 universal groups, and in [9], where a complete list of all such graphs on up to 768 vertices was compiled.

The purpose of this paper is to begin an investigation of the next interesting case where $\{d_1, d_2\} = \{3, 4\}$. In Section 3, we show that in this case the edge kernel $G_{uv}^{[1]}$ can be arbitrary large. This is in sharp contrast with the behaviour of locally G -arc-transitive graphs of valence $\{3, 3\}$, where the order of $G_{uv}^{[1]}$ divides 32 (see [9, 16]).

In the study of locally G -arc-transitive graphs, the case where the edge kernel $G_{uv}^{[1]}$ is not trivial is rather special (see for example [5]). In this paper, we shall restrict ourselves to the locally G -arc-transitive graphs of valence $\{3, 4\}$ with the trivial edge kernel. We will prove that, in this case, the group G is a quotient of one of nineteen *universal* infinite finitely presented groups (that we give explicitly in terms of generators and relators).

Theorem 1. *Let Γ be a connected locally G -arc-transitive graph of valence $\{3, 4\}$ and let uv be an edge of Γ with $\text{val}(v) = 3$ and $\text{val}(u) = 4$. If $G_{uv}^{[1]} = 1$, then for some $i \in \{0, 1, \dots, 18\}$ there exists an epimorphism from the group U_i , given in Table 1, onto G , which maps the subgroups L_i , B_i and R_i isomorphically onto G_v , G_{uv} and G_u , respectively.*

Table 1: Universal groups for locally arc-transitive graphs of valence $\{3, 4\}$ with $G_{uv}^{[1]} = 1$.

$U_0 = \langle a, c \mid a^3, c^4 \rangle$ $L_0 = \langle a \rangle \cong C_3, \quad B_0 = 1, \quad R_0 = \langle c \rangle \cong C_4$
$U_1 = \langle a, x, y \mid a^3, x^2, y^2, [x, y] \rangle$ $L_1 = \langle a \rangle \cong C_3, \quad B_1 = 1, \quad R_1 = \langle x, y \rangle \cong C_2 \times C_2$
$U_2 = \langle a, b, c \mid a^3, b^2, c^4, [a, b], (bc)^2 \rangle$ $L_2 = \langle a, b \rangle \cong C_6, \quad B_2 = \langle b \rangle \cong C_2, \quad R_2 = \langle b, c \rangle \cong D_4$
$U_3 = \langle a, b, c \mid a^3, b^2, c^4, (ab)^2, (bc)^2 \rangle$ $L_3 = \langle a, b \rangle \cong S_3, \quad B_3 = \langle b \rangle \cong C_2, \quad R_3 = \langle b, c \rangle \cong D_4$
$U_4 = \langle a, c, x, y \mid a^9, c^3, x^2, y^2, a^3 = c, [x, y], x^c = y, y^c = xy \rangle$ $L_4 = \langle a, c \rangle \cong C_9, \quad B_4 = \langle c \rangle \cong C_3, \quad R_4 = \langle c, x, y \rangle \cong A_4$

$U_5 = \langle a, c, x, y \mid a^3, c^3, x^2, y^2, [a, c], [x, y], x^c = y, y^c = xy \rangle$ $L_5 = \langle a, c \rangle \cong C_3 \times C_3, \quad B_5 = \langle c \rangle \cong C_3, \quad R_5 = \langle c, x, y \rangle \cong A_4$
$U_6 = \langle a, b, c, x, y \mid a^9, b^2, c^3, x^2, y^2, a^3 = c, (ab)^2, [x, y], (bc)^2, x^c = y, y^c = xy, x^b = y \rangle$ $L_6 = \langle a, b, c \rangle \cong D_9, \quad B_6 = \langle b, c \rangle \cong S_3, \quad R_6 = \langle b, c, x, y \rangle \cong S_4$
$U_7 = \langle a, b, c, x, y \mid a^3, b^2, c^3, x^2, y^2, [x, y], (bc)^2, x^c = y, y^c = xy, x^b = y, [a, c], (ab)^2 \rangle$ $L_7 = \langle a, b, c \rangle \cong \text{GenDih}(C_3 \times C_3), \quad B_7 = \langle b, c \rangle \cong S_3, \quad R_7 = \langle b, c, x, y \rangle \cong S_4$
$U_8 = \langle a, b, c, x, y \mid a^3, b^2, c^3, x^2, y^2, [x, y], (bc)^2, x^c = y, y^c = xy, x^b = y, [a, c], [a, b] \rangle$ $L_8 = \langle a, b, c \rangle = \langle c \rangle \rtimes \langle a, b \rangle \cong C_3 \rtimes C_6, \quad B_8 = \langle b, c \rangle \cong S_3, \quad R_8 = \langle b, c, x, y \rangle \cong S_4$
$U_9 = \langle c, d, x, y \mid c^3, d^2, x^2, y^2, (cd)^2, [d, x], [d, y], [x, y] \rangle$ $L_9 = \langle c, d \rangle \cong S_3, \quad B_9 = \langle d \rangle \cong C_2, \quad R_9 = \langle d, x, y \rangle \cong C_2 \times C_2 \times C_2$
$U_{10} = \langle c, d, x \mid c^3, d^2, x^4, (cd)^2, [x, d] \rangle$ $L_{10} = \langle c, d \rangle \cong S_3, \quad B_{10} = \langle d \rangle \cong C_2, \quad R_{10} = \langle d, x \rangle \cong C_2 \times C_4$
$U_{11} = \langle c, d, x, y \mid c^3, d^2, x^4, y^2, (cd)^2, x^2 = d, [x, y] \rangle$ $L_{11} = \langle c, d \rangle \cong S_3, \quad B_{11} = \langle d \rangle \cong C_2, \quad R_{11} = \langle d, x, y \rangle \cong C_2 \times C_4$
$U_{12} = \langle c, d, x \mid c^3, d^2, x^8, (cd)^2, x^4 = d \rangle$ $L_{12} = \langle c, d \rangle \cong S_3, \quad B_{12} = \langle d \rangle \cong C_2, \quad R_{12} = \langle d, x \rangle \cong C_8$
$U_{13} = \langle c, d, x, y \mid c^3, d^2, x^4, y^2, (cd)^2, x^2 = d, (xy)^2 \rangle$ $L_{13} = \langle c, d \rangle \cong S_3, \quad B_{13} = \langle d \rangle \cong C_2, \quad R_{13} = \langle d, x, y \rangle \cong D_4$
$U_{14} = \langle c, d, x, y \mid c^3, d^2, x^4, y^4, (cd)^2, x^2 = y^2 = [x, y] = d \rangle$ $L_{14} = \langle c, d \rangle \cong S_3, \quad B_{14} = \langle d \rangle \cong C_2, \quad R_{14} = \langle d, x, y \rangle \cong Q$
$U_{15} = \langle a, c, d, x, y \mid a^3, c^3, d^2, x^2, y^2, (dc)^2, [a, c], [a, d], [d, x], [d, y], [x, y], x^a = y, y^a = xy \rangle$ $L_{15} = \langle a, c, d \rangle \cong C_3 \times S_3, \quad B_{15} = \langle a, d \rangle \cong C_6, \quad R_{15} = \langle a, d, x, y \rangle \cong C_2 \times A_4$
$U_{16} = \langle a, c, d, x, y \mid a^3, c^3, d^2, x^4, y^4, (dc)^2, [a, c], [a, d], x^2 = y^2 = [x, y] = d, x^a = y, y^a = xy \rangle$ $L_{16} = \langle a, c, d \rangle \cong C_3 \times S_3, \quad B_{16} = \langle a, d \rangle \cong C_6, \quad R_{16} = \langle a, d, x, y \rangle \cong Q \rtimes C_3$
$U_{17} = \langle a, b, c, d, x, y \mid a^3, b^2, c^3, d^2, x^2, y^2, (ba)^2, (dc)^2, [a, c], [a, d], [b, c], [b, d], [x, d], [y, d], [x, y], x^a = y, y^a = xy, x^b = x, y^b = xy \rangle$ $L_{17} = \langle a, b, c, d \rangle = S_3 \times S_3, \quad B_{17} = \langle a, b, d \rangle \cong C_2 \times S_3, \quad R_{17} = \langle a, b, d, x, y \rangle \cong C_2 \times S_4$
$U_{18} = \langle a, b, c, d, x, y \mid a^3, b^2, c^3, d^2, x^4, y^4, (ba)^2, (dc)^2, [a, c], [a, d], [b, c], [b, d], x^2 = y^2 = [x, y] = d, x^a = y, y^a = xy, x^b = x^{-1}, y^b = yx \rangle$ $L_{18} = \langle a, b, c, d \rangle = S_3 \times S_3, \quad B_{18} = \langle a, b, d \rangle \cong C_2 \times S_3, \quad R_{18} = \langle a, b, d, x, y \rangle \cong Q \rtimes S_3$

As an application of Theorem 1, we compute a complete list of all connected biregular graphs of valence $\{3, 4\}$ on at most 350 vertices (and thus at most 600 edges) admitting a locally arc-transitive group of automorphisms G with $G_{uv}^{[1]} = 1$. This was done by applying the `LowIndexNormalSubgroups` routine, implemented in `Magma` [6], to find, for each of the groups U_i , all the quotients G of U_i of order at most $600|B_i|$ such that the group L_i and R_i project isomorphically onto some subgroups L and R of G . Once such triples (G, L, R) were obtained, we constructed for each of them the so called *coset graph* $\text{Cos}(G, L, R)$, the vertex set of which is the disjoint union $G/L \cup G/R$ of coset sets with edges of the form $\{Lg, Rg\}$ for $g \in G$ (see, for example, [15] for details). In this way, we found 220 pairwise non-isomorphic graphs. A complete list of graphs (in magma code) can be accessed at [23]. In Section 4, we present some graph-theoretical parameters of the 42 graphs from that list with at most 100 vertices.

Let us also mention that a triple of finite groups (L, B, R) with $B = L \cap R$ (such as a triple (L_i, B_i, R_i) from Table 1) is often called a *finite group amalgam* of index $([L : B], [R : B])$ (that is, of index $(3, 4)$ in the case of the amalgams (L_i, B_i, R_i) from Table 1). Note that the edge-stabiliser G_{uv} in a connected locally G -arc-transitive graph is core-free in G (that is, contains no nontrivial subgroups that are normal in G). This implies that the amalgams (L, B, R) (where $L = G_v$, $B = G_{uv}$ and $R = G_u$) arising from such pairs (Γ, G) have the following property: if N is a subgroup of G_{uv} which is normal both in G_v and G_u , then N is trivial. Such amalgams are often called *faithful*. Furthermore, the requirement that the edge kernel $G_{uv}^{[1]}$ is trivial translates into the requirement that the intersection $\text{core}_L(B) \cap \text{core}_R(B)$ of the cores of B in L and R is trivial. We say that such amalgams have a *trivial edge kernel*. With this terminology in mind, Theorem 1 can thus also be viewed as a classification of faithful finite group amalgams of index $\{3, 4\}$ with trivial edge kernel. (We refer the reader to [19] for further details on the relationship between amalgams and locally arc-transitive graphs.)

Corollary 2. *If (L, B, R) is a finite faithful amalgam of index $\{3, 4\}$ with trivial edge kernel, then it is isomorphic to one of the amalgams (L_i, B_i, R_i) given in Table 1.*

We prove Theorem 1 in Section 2. In Section 3, we present a construction (and characterisation) of locally arc-transitive graphs with arbitrary large kernel. Finally, in Section 4, a list of all connected graphs of valence $\{3, 4\}$ of order up to 100 that admit a locally arc-transitive action of a group with a trivial edge kernel, is given and several graph theoretical parameters of these graphs are computed.

2. PROOF OF THEOREM 1

Our approach to the proof of Theorem 1 can be summarised as follows: Since $G_{uv}^{[1]} = 1$, it follows that G_{uv} acts faithfully on $\Gamma(uv)$ and is thus a subgroup of $S_3 \times C_2$. This gives only finitely possibilities for G_{uv} as an abstract group. Furthermore, since G_{uv} is embedded in G_u and G_v as subgroups of index 4 and 3, respectively, there is only finitely many possible embeddings of G_{uv} into G_u and G_v . For each possible pair of embeddings, we shall write the groups G_u and G_v as finitely presented groups with the common generators generating the group G_{uv} . Finally, since the group G is generated by G_u and G_v , this will allow us to conclude that G is a quotient of the group generated by the union of the generators of G_u

and G_v subject to the union of relators of G_u and G_v , respectively. We refer the reader to [21] for the details of this procedure.

To simplify notation, let $K_v = G_v^{[1]}$ and $K_u = G_u^{[1]}$. Since $G_{uv}^{[1]} = 1$, it follows that the mapping which maps an element $g \in G_{uv}$ to the permutation \bar{g} induced by the action of g on $\Gamma(uv)$ is an isomorphism of groups G_{uv} and $G_{uv}^{\Gamma(uv)}$. Being a kernel of group epimorphisms, the groups K_u and K_v are isomorphic to some normal subgroups of $\text{Sym}(\Gamma(u) \setminus \{v\}) \cong C_2$ and $\text{Sym}(\Gamma(v) \setminus \{u\}) \cong S_3$, respectively. This shows that K_u is either trivial or isomorphic to C_2 , while K_v is either trivial, isomorphic to C_3 , or to S_3 . We will split the proof into two cases, depending on whether K_u is trivial or not.

Case A Suppose that $K_u = 1$. Then, of course, G_u acts faithfully and transitively on $\Gamma(u)$ and $G_u \cong G_u^{\Gamma(u)} \leq S_4$. The group G_v is therefore isomorphic to one of the groups C_4 , $C_2 \times C_2$, D_4 , A_4 or S_4 .

Case A.1 Let us first deal with the case where G_u is isomorphic to C_4 or $C_2 \times C_2$. Then $G_{uv} = 1$ and thus $K_v = 1$, implying that $G_v \cong G_v^{\Gamma(v)}$ is a regular subgroup of S_3 ; in particular, $G_v \cong C_3$. Since G is generated by G_u and G_v , this implies that G is a quotient of the free product $U_0 \cong C_3 * C_4$ or of the free product $U_1 \cong C_3 * (C_2 \times C_2)$, with $G_v \cong L_i$, $G_u \cong R_i$ and $G_{uv} \cong B_i$ for $i = 0$ or 1 .

Case A.2 Suppose now $G_u \cong D_4$. Let $b \in G_u$ be the involution fixing v and let $c \in G_u$ be an element that cyclically permutes the neighbours of u . Note that $G_u = \langle b, c \mid b^2, c^4, (bc)^2 \rangle$ and $G_{uv} = \langle b \rangle$. Furthermore, since $|G_v| = 3|G_{uv}| = 6$, it follows that $G_v \cong C_6$ or S_3 . In both cases, let a be a generator of the unique subgroup of order 3 in G_v . Then $G_v = \langle a, b \rangle$ and we see that $[a, b] = 1$ if $G_v \cong C_6$ and $(ab)^2 = 1$ if $G_v \cong S_3$. Since G is generated by G_v and G_u , it follows that G is a quotient of the group U_2 (if $G_v \cong C_6$) or of U_3 if $(G_v \cong S_3)$.

Case A.3 Suppose finally that $G_u \cong A_4$ or S_4 . Let x and y be the generators of the regular normal subgroup of G_u , isomorphic to $C_2 \times C_2$. Observe that $G_u = \langle x, y \rangle \rtimes G_{uv}$ where $G_{uv} \cong C_3$ (if $G_u \cong A_4$) or $G_{uv} \cong S_3$ (if $G_u \cong S_4$). Note also that the action of G_{uv} on $\Gamma(v) \setminus \{u\}$ is equivalent to the action of G_{uv} by conjugation on the nontrivial elements of $\langle x, y \rangle$. Let c be the element of order 3 in G_{uv} that cyclically permutes the elements x, y, xy in that order, and if $G_u \cong S_4$, let b be the involution of G_{uv} for which $y = x^b$. The generators x, y, c (and possibly b) of G_v then satisfy the relations:

$$x^2 = y^2 = [x, y] = c^3 = b^2 = 1, \quad x^c = y, y^c = xy, x^b = y, (cb)^2 = 1.$$

Note that since c is of order 3, it acts trivially on $\Gamma(v) \setminus \{u\}$; that is $c \in K_v$. Moreover, since $\langle c \rangle$ is characteristic in G_{uv} , it follows that $\langle c \rangle$ is normal in G_v .

If $G_u \cong A_4$ (and thus $G_{uv} = \langle c \rangle = K_v$), then $|G_v| = 3|G_{uv}| = 9$ and therefore $G_v \cong C_9$ or $C_3 \times C_3$. In the former case, we see that G_v is generated by some element, say a , of order 9 satisfying $a^3 = c$. Since G is generated by G_v and G_u , it must be a quotient of the group of the group U_4 in Table 1. On the other hand, if $G_v \cong C_3 \times C_3$, then it is generated by c and some element a of order 3 satisfying $[a, c] = 1$, and G is a quotient of the group U_5 in Table 1.

Suppose now that $G_u \cong S_4$ (and thus $G_{uv} = \langle b, c \rangle$). Then either $K_v = G_{uv}$, implying that $G_v^{\Gamma(v)} \cong C_3$, or $K_v = \langle c \rangle$ and thus $G_v^{\Gamma(v)} \cong S_3$. As observed above, in both cases $\langle c \rangle$ is normal in G_v . Consider the Sylow 3-subgroup P of G_v . Since $|G_v| = 3|G_{uv}| = 18$, it follows that P is normal of index 2 in G_v and is isomorphic

either to $C_3 \times C_3$ or to C_9 . Since G_v contains an involution b , it follows that $G_v = P \rtimes \langle b \rangle$.

If $P = C_9$, then let a be its generator such that $a^3 = c$. Since G_v contains $G_{uv} \cong S_3$ and is thus nonabelian, it follows that $G_v = \langle a, b \rangle \cong D_9$ (note that D_9 is the only nonabelian group of order 18 containing a cyclic subgroup of order 9). In particular, G_v is generated by a and b , subject to relations $a^9 = b^2 = (ab) = 1$. The group G is then a quotient of the group U_6 in Table 1.

If $P = C_3 \times C_3$, then consider the action by conjugation of $\langle b \rangle \cong C_2$ on the three complements of $\langle c \rangle$ in P . This action has at least one fixed point, say $\langle a \rangle \cong C_3$. The two possibilities for the action of b by conjugation on $\langle a \rangle$ give rise to two possibilities for the stabiliser G_v :

$$\begin{aligned} G_v &= \langle a, b, c \mid a^3, c^3, b^2, [a, c], (bc)^2, [a, b] \rangle = \langle c \rangle \rtimes \langle a, b \rangle \cong C_3 \rtimes C_6; \\ G_v &= \langle a, b, c \mid a^3, c^3, b^2, [a, c], (bc)^2, (ab)^2 \rangle \cong \text{GenDih}(C_3 \times C_3). \end{aligned}$$

The group G is thus a quotient of the group U_7 or the group U_8 in Table 1.

Case B Suppose now that $K_u \neq 1$, and thus $K_u \cong C_2$. Let d be the generator of K_u . Since K_u acts transitively on the set $\Gamma(v) \setminus \{u\}$, and since K_v is equal to the point-stabiliser of the action of the group G_{uv} on $\Gamma(v) \setminus \{u\}$, it follows that

$$G_{uv} = \langle K_v, K_u \rangle = K_v \times K_u = K_v \times \langle d \rangle \cong K_v \times C_2.$$

Since G_{uv} contains d , it follows that $G_v^{\Gamma(v)} \cong S_3$. Moreover, in view of the isomorphism $G_{uv} \cong G_{uv}^{\Gamma(uv) \setminus \{u, v\}}$, it follows that K_v is isomorphic to a normal subgroup of $\text{Sym}(\Gamma(u) \setminus \{v\}) \cong S_3$. We thus need to consider the cases $K_v = 1$, $K_v \cong C_3$ and $K_v \cong S_3$.

Case B.1 Suppose first that $K_v = 1$. Then $G_v \cong G_v^{\Gamma(v)}$ and hence $G_v = \langle c, d \rangle \cong S_3$, where c is an arbitrary element of G_v of order 3 (and thus satisfying the relation $(cd)^2 = 1$). On the other hand, $|G_u| = 4|G_{uv}| = 8$, implying that G_u is one of the five groups of order 8 (the three abelian groups of order 8, the dihedral group D_4 or the quaternion group Q). Note that apart from $C_4 \times C_2$, in the remaining four groups of order 8, the central involutions are conjugate under the automorphism group of the group, implying that there is essentially a unique way how to embed d into any of these groups. On the other hand, there are two types of involutions in $C_4 \times C_2$, those that are squares of elements of order 4 and those that are not. Here the element d can be embedded in two essentially distinct ways. This therefore gives rise to six possible stabilisers G_u , listed below, each generating together with $G_v = \langle c, d \rangle$ a quotient of one of the groups U_9, \dots, U_{14} in Table 1.

$$\begin{aligned} G_u &= \langle d, x, y \mid d^2, x^2, y^2, [d, x], [d, y], [x, y] \rangle \cong C_2 \times C_2 \times C_2; \\ G_u &= \langle d, x \mid d^2, x^4, [x, d] \rangle \cong C_4 \times C_2; \\ G_u &= \langle d, x, y \mid d^2, x^4, y^2, x^2 = d, [x, y] \rangle \cong C_4 \times C_2; \\ G_u &= \langle d, x \mid d^2, x^8, x^4 = d \rangle \cong C_8; \\ G_u &= \langle d, x, y \mid d^2, x^4, y^2, x^2 = d, (xy)^2 \rangle \cong D_4; \\ G_u &= \langle d, x, y \mid d^2, x^4, y^4, x^2 = y^2 = [x, y] = d \rangle \cong Q. \end{aligned}$$

Case B.2 Suppose now that $K_v \cong C_3$. Then $K_v = \langle a \rangle$ for some element a of order 3 which cyclically permutes the elements of $\Gamma(u) \setminus \{v\}$. Then

$$G_{uv} = K_v \times K_u = \langle a, d \rangle \cong C_6.$$

Moreover, since G_{uv} is transitive both on $\Gamma(u) \setminus \{v\}$ as well as on $\Gamma(v) \setminus \{u\}$, it follows that $G_u^{\Gamma(u)}$ and $G_v^{\Gamma(v)}$ are doubly transitive groups and hence Γ is locally $(G, 2)$ -arc-transitive. In particular, $G_v^{\Gamma(u)} \cong S_3$ and since $G_{uv}^{\Gamma(u)} \cong C_3$, also $G_u^{\Gamma(u)} \cong A_4$.

Let us now determine the structure of the group G_v . Observe first that $|G_v| = 3|G_{uv}| = 18$ and let P be a Sylow 3-subgroup of G_v . Since $[G_v : P] = 2$, we see that P is normal in G_v and therefore $a \in P$. Moreover, since $d \in G_v \setminus P$, the group G_v splits over P into a semidirect product $P \rtimes \langle d \rangle$. Now consider the action of $\langle d \rangle$ upon P by conjugation. If this action is trivial, then G_v is abelian and $K_u = \langle d \rangle$ is normal in both G_v and G_u . But then K_u is normal in G , implying that K_u acts trivially on the set of edges of the graph, which is clearly a contradiction.

If $P \cong C_9$, then $\text{Aut}(P)$ contains a unique involution, namely the one inverting the elements of P . Hence $G_v \cong D_9$. But since D_9 contains no subgroup isomorphic to $C_6 \cong G_{uv}$, this cannot occur in this case.

Therefore $P \cong C_3 \times C_3$. Now, similarly as in the last paragraph of Case A.3, consider the action of $\langle d \rangle$ on the set of the four subgroups of P of order 3 by conjugation. Since $\langle d \rangle$ has order 2 and already fixes one such group (namely the group $\langle a \rangle$), it must fix at least one other; let c be its generator. Since d does not centralise P , but centralises a , it follows that $[c, d] \neq 1$, and thus $c^d = c^{-1}$ (or equivalently $(cd)^2 = 1$). This shows that

$$(1) \quad G_v = \langle a, c, d \rangle \text{ where } c^3 = a^3 = [a, c] = 1, d^2 = [d, a] = (cd)^2 = 1.$$

Note that $G_v = \langle a \rangle \times \langle c, d \rangle \cong C_3 \times S_3$.

Let us now consider the structure of G_u . Let $\pi: G_u \rightarrow G_u^{\Gamma(u)}$ be the epimorphism that maps each element $g \in G_u$ to the permutation induced by g on $\Gamma(u)$. Note that the kernel of π is $K_u = \langle d \rangle$ and in particular that $\langle d \rangle$ is normal (and therefore central) in G_u . Recall that $G_u^{\Gamma(u)} \cong A_4$, let V be the regular normal subgroup of $G_u^{\Gamma(u)}$, isomorphic to the Klein group $C_2 \times C_2$, and let $P = \pi^{-1}(V)$. Then P is a normal Sylow 2-subgroup of G_u , implying that $d \in P$ and $G_u = P \rtimes \langle a \rangle$. Since $\langle a \rangle$ is normal in G_v , it is not normal in G_u (since otherwise it would act trivially on the edge set of Γ), implying that the action of $\langle a \rangle$ upon P by conjugation is nontrivial. Note that the only groups of order 8 that admit an automorphism of order 3 are the elementary abelian group C_2^3 and the quaternion group Q . This leaves us with two possibilities: $P \cong C_2^3$ and $P \cong Q$.

B.2.1 Suppose that $P \cong C_2^3$. Then the fact that a centralises d implies that $\langle a \rangle$ fixes one of the four complements of $\langle d \rangle$ in P . Clearly we may choose the generators x and y of that complement in such a way that $x^a = y$ and $y^a = xy$. We have thus shown that G_u is generated by elements x, y, d, a , which, in addition to relations in (1), satisfy also the following:

$$(2) \quad x^2 = y^2 = [x, d] = [y, d] = [x, y] = 1, x^a = y, y^a = xy.$$

The group $G = \langle G_u, G_v \rangle$ must thus be a quotient of the group U_{15} in Table 1. Observe also that $G_u \cong \langle d \rangle \times \langle x, y, a \rangle \cong C_2 \times A_4$.

B.2.2 Suppose now that P is isomorphic to the quaternion group Q . Since the centre of Q is of order 2 and since d is central in P , it follows that $Z(P) = \langle d \rangle$.

Furthermore, since a centralises d , the partition $\{\{g, gd\} : g \in P \setminus \{1, d\}\}$ of the set $P \setminus \{1, d\}$ into the nontrivial cosets of $\langle d \rangle$ is $\langle a \rangle$ -invariant. Hence the action of $\langle a \rangle$ on P by conjugation gives rise to two orbits on $\{\{g, gd\} : g \in P \setminus \{1, d\}\}$, each intersecting each pair $\{g, gd\}$ in a unique point. Moreover, since $P/\langle d \rangle \cong C_2 \times C_2$, if $\{x, xd\}$ and $\{y, yd\}$ are distinct nontrivial cosets, then $\{xy, xyd\}$ is the third one. Now choose $x_0 \in P \setminus \{1, d\}$ and let $y_0 = x_0^a$. Then $y_0^a \in \{x_0 y_0, x_0 y_0 d\}$. If $y_0^a = x_0 y_0$, then let $x = x_0$ and $y = y_0$; otherwise, let $x = x_0 d$ and $y = y_0 d$. Observe that in both cases it follows that $x^a = y$ and $y^b = xy$. We have thus shown that G_u is generated by elements x, y, d, a , which, in addition to relations in (1), satisfy also the following:

$$(3) \quad x^4 = y^4 = 1, x^2 = y^2 = [x, y] = d, x^a = y, y^a = xy.$$

The group $G = \langle G_u, G_v \rangle$ is therefore a quotient of the group U_{16} in Table 1.

Case B.3 Suppose finally that $K_v \cong S_3$. Then K_v is generated by some elements a, b satisfying $a^3 = b^2 = (ab)^2 = 1$ and thus

$$(4) \quad G_{uv} = K_v \times K_u = \langle a, b, d \rangle \cong S_3 \times C_2.$$

Let us first determine the structure of the vertex-stabiliser G_v . Since the centre and the outer automorphism group of $K_v \cong S_3$ are both trivial, it is not difficult to see that K_v must be a direct factor in every group in which it is normal (see for example [24, 13.5.8 and Exercise 13.5]). In particular, $G_v = K_v \times H$ where $H \cong H^{\Gamma(v)} = G_v^{\Gamma(v)} \cong S_3$. Now consider the group $H_u = H \cap G_u = H \cap G_{uv}$. Since H is normal in G_v , it follows that H_u is normal in $G_v \cap G_u = G_{uv}$. However, since $H \cong H^{\Gamma(v)} \cong S_3$, we see that $H_u \cong S_2$, implying that H_u is central in G_{uv} . However, the centre of $G_{uv} = \langle a, b, d \rangle$ is the group $\langle d \rangle$, showing that $H_u = \langle d \rangle = K_u$.

If we let c be an element of order 3 in $H \cong S_3$, then $H = \langle c, d \rangle$, with $(dc)^2 = 1$. Note that c commutes with a and b (since $c \in H$ and $G_v = K_v \times H$). This implies that G_v is generated by elements a, b, c, d , subject to relations

$$(5) \quad a^3 = b^2 = c^3 = d^2 = (ba)^2 = (dc)^2 = [a, c] = [a, d] = [b, c] = [b, d] = 1.$$

In particular, $G_v = \langle a, b \rangle \times \langle c, d \rangle \cong S_3 \times S_3$ is isomorphic to the group L_4 in Table 1.

We will now determine the structure of the group G_u . Observe first that $|G_u| = 4|G_{uv}| = 48$. Since $|K_u| = 2$, it follows that $|G_u^{\Gamma(u)}| = 24$ and hence $G_u^{\Gamma(u)} \cong S_4$. Note also that since $\langle d \rangle = K_u$ is a normal subgroup of G_u of order 2, the element d is central in G_u . As in Case B.2.1, let $\pi: G_u \rightarrow G_u^{\Gamma(u)}$ denote the group epimorphism that maps an element $g \in G_u$ to the permutation induced by the action of g on $\Gamma(u)$.

Further, let V be the regular normal subgroup of $G_u^{\Gamma(u)}$, isomorphic to the Klein group $C_2 \times C_2$, and let $P = \pi^{-1}(V)$ be its preimage in G_u . Then P is normal in G_u and has order 8. Moreover,

$$P/\langle d \rangle \cong V \text{ and } G_u/P \cong G_u^{\Gamma(u)}/V \cong S_3.$$

Since V is transitive on $\Gamma(u)$, so is P , implying that $G_u = PG_{uv}$. However, $G_{uv} = K_u K_v$ and $K_u \leq P$, showing that $G_u = PK_v$. Since $|G_u| = 48 = 8 \cdot 6 = |P| |K_v|$, it follows that $P \cap K_v = 1$ and thus

$$(6) \quad G_u = P \rtimes K_v.$$

Let us now consider the action of $K_v = \langle a, b \rangle \cong S_3$ on P by conjugation. If the kernel of this action is nontrivial, then it contains a (since $\langle a \rangle$ is the unique minimal

normal subgroup of K_v). In this case, G_u contains a subgroup isomorphic to $P \times C_3$, implying that $G_u^{\Gamma(u)}$ contains a subgroup isomorphic to $(P \times C_3)/\langle d \rangle \cong V \times C_3$. This contradicts the fact that $G_u^{\Gamma(u)}$ is isomorphic to S_4 , and thus shows that $K_v \cong S_3$ embeds into $\text{Aut}(P)$. Out of five groups of order 8 (the three abelian ones, the dihedral group D_4 and the quaternion group) only the elementary abelian group of order 8 and the quaternion group admit S_3 as a group of automorphisms.

Case B.3.1 Suppose P is elementary abelian. Then we proceed similarly as in Case B.2.1. Since K_v centralises d , it permutes (via conjugation) the set of 4 complements of $\langle d \rangle$ in P . Since every action of S_3 on 4 points has at least one fixed point, this implies that K_v normalises at least one complement of $\langle d \rangle$ in P , say W . Since K_v acts trivially on $\langle d \rangle$, it must act faithfully as a group of automorphisms of W . But since $\text{Aut}(W) = \text{GL}(2, 3) \cong S_3$, the action of K_v on W is uniquely determined. In particular, we may choose generators x and y of W in such a way that $x^a = y$, $y^a = xy$, $x^b = x$ and $y^b = xy$.

We have thus shown that G_u is generated by elements x, y, d, a, b , which, in addition to relations in (5), satisfy also the following:

$$(7) \quad x^2 = y^2 = [x, d] = [y, d] = [x, y] = 1, x^a = y, y^a = xy, x^b = x, y^b = xy.$$

Note that $G_u = \langle d, x, y \rangle \rtimes \langle a, b \rangle \cong (C_2^3) \rtimes S_3$, as well as $G_u = \langle d \rangle \times \langle x, y, a, b \rangle \cong C_2 \times S_4$. In particular, G_u is isomorphic to the group R_{17} in Table 1. Finally, since G is generated by G_v and G_u , formulas (5) and (7) show that G is a quotient of the group U_{17} in Table 1.

Case B.3.2 Suppose now that P is the quaternion group. This case is analogous to Case B.2.2. Since the centre of P is of order 2 and since d is central in P , it follows that $Z(P) = \langle d \rangle$. Furthermore, since K_v centralises d , the partition $\{\{g, gd\} : g \in P \setminus \{1, d\}\}$ of the set $P \setminus \{1, d\}$ into the nontrivial cosets of $\langle d \rangle$ is K_v -invariant. Hence the action of $K_v = \langle a \rangle$ on P by conjugation gives rise two orbits on $\{\{g, gd\} : g \in P \setminus \{1, d\}\}$, each intersecting each pair $\{g, gd\}$ in a unique point. Moreover, since $P/\langle d \rangle \cong C_2 \times C_2$, if $\{x, xd\}$ and $\{y, yd\}$ are distinct nontrivial cosets, then $\{xy, xyd\}$ is the third one. Clearly b fixes at least one coset setwise, say $\{x_0, x_0d\} \subseteq P \setminus \{1, d\}$. Let $y_0 = x_0^a$ and note that $y_0^a \in \{x_0y_0, x_0y_0d\}$. If $y_0^a = x_0y_0$, then let $x = x_0$ and $y = y_0$; otherwise, let $x = x_0d$ and $y = y_0d$. In both cases we see that $x^a = y$. Moreover, in the former case, we clearly have also $y^a = xy$. However, even in the latter case, we see that $y^a = (y_0d)^a = y_0^ad = (x_0y_0d)d = (x_0d)(y_0d) = xy$. Hence in both cases we see that $y^a = xy$. On the other hand, since we have chosen x_0 in such a way that b fixes $\{x_0, x_0d\}$ setwise, we have either $x^b = x$ or $x^b = xd$. If $x^b = x$, then $y^b = x^{ab} = (x^b)^{a^{-1}} = xy$, and since b is an involution, also $(xy)^b = y$. But then $y = (xy)^b = (dyx)^b = d^by^bx^b = dxyx = yx^2 = yd$. This contradiction shows that $x^b = xd = x^{-1}$, and consequently, $y^b = x^{ab} = (x^b)^{a^{-1}} = (xy)^{-1} = xyd = yx$. Since any two generators x, y of the quaternion group satisfy the relations $x^4 = y^4 = 1$, $x^2 = y^2 = [x, y] = d$ (where d is the central involution), it follows that G_u is generated by elements x, y, d, a, b which, in addition to (5) satisfy also the relations:

$$(8) \quad x^4 = y^4 = 1, x^2 = y^2 = [x, y] = d, x^a = y, x^b = x^{-1}, y^a = xy, y^b = yx.$$

Since G is generated by G_v and G_u , formulas (5) and (8) show that G is a quotient of the group U_{18} in Table 1. This completes the proof of Theorem 1.

3. LOCALLY ARC-TRANSITIVE GROUP ACTIONS WITH LARGE EDGE KERNEL

In this section we first show that the edge kernel in a locally arc-transitive graph of valence $\{3, 4\}$ can be arbitrary large. This will be proved by means of the so called *subdivided doubles*, a construction that was introduced in [22] and that can be described as follows.

Let Λ be a G -arc-transitive k -valent graph. The *subdivision* of Λ , denoted by $\mathbb{S}\Lambda$, is the bipartite graph of valence $\{2, k\}$ with vertex set $V(\Lambda) \cup E(\Lambda)$ in which each $e \in E(\Lambda)$ is adjacent to the endpoints of e in Λ . The *subdivided double* of Λ is then the graph obtained from $\mathbb{S}\Lambda$ by blowing up each original vertex $v \in V(\Lambda)$ to a pair of vertices, each being adjacent to the neighbours of v in $\mathbb{S}\Lambda$. More precisely, the vertex set of $\mathbb{D}_2\Lambda$ can be defined as the disjoint union $(\mathbb{Z}_2 \times V(\Lambda)) \cup E(\Lambda)$ with edges of the form $(i, v)e$ for any $i \in \mathbb{Z}_2$, $v \in V(\Lambda)$ and $e \in E(\Lambda)$ such that v is an endpoint of e . Note that for any $v \in V(\Lambda)$, the permutation that interchanges the vertices $(0, v)$ and $(1, v)$ in $\mathbb{D}_2\Lambda$ and fixes all other vertices of $\mathbb{D}_2\Lambda$ is an automorphism of $\mathbb{D}_2\Lambda$. The group generated by all such automorphisms is elementary abelian of order $2^{|V(\Lambda)|}$. Together with the group induced by the obvious action of G on $\mathbb{D}_2\Lambda$, this group generates a group which acts locally arc-transitively on $\mathbb{D}_2\Lambda$ and has a large edge kernel. This can be summarised as follows (the proof can be found in [22, Lemma 4.2]).

Lemma 3. *If Λ is arc-transitive graph of valence k and with n vertices, then $\mathbb{D}_2\Lambda$ is a locally G -arc-transitive graph of valence $\{2, k\}$ for some group G with the edge kernel $G_{uv}^{[1]}$ of order divisible by 2^{n-2} .*

Following [22, Definition 4.1] we shall call a graph *unworthy* if two of its vertices share the same neighbourhood. The following lemma is a converse of Lemma 3 in the context of locally G -arc-transitive graphs of valence $\{3, 4\}$.

Lemma 4. *Let Γ be a connected unworthy locally G -arc-transitive graph of valence $\{3, 4\}$. Then Γ is isomorphic either to the complete bipartite graph $K_{3,4}$ or to the subdivided double $\mathbb{D}_2\Lambda$ for some connected cubic G -arc-transitive graph Λ .*

Proof. For every vertex u of Γ , let $B(u)$ be the set of all vertices of Γ that have the same neighbourhood as u . Since Γ is unworthy, we see that for some vertex u , the size of $B(u)$ is at least 2; let U be the part of the bipartition of Γ containing u and let $W = V(\Gamma) \setminus U$.

Since Γ is locally G -arc-transitive, the group G acts transitively on U as well as on W and the set $\mathcal{P} = \{B(v) : v \in U\}$ is a G -invariant partition of U . In particular, every $w \in W$ is adjacent to a fixed number of blocks in \mathcal{P} . Observe also that for any given $B \in \mathcal{P}$ and $w \in W$, the vertex w is adjacent either to none or to all of the vertices in B . In particular, the size of B is a divisor of the valence of w . Since $|B| \geq 2$, this shows that either the valence of w is 4 and $|B| = 2$, or the valence of w equals the size of B . In the latter case each vertex of W is adjacent to precisely one block in \mathcal{P} , which, together with connectivity of Γ , implies that $\Gamma \cong K_{3,4}$.

We may thus assume that $|B| = 2$ and that the valence of w is 4 and that no two vertices of valence 4 share the same neighbourhood. Now consider the bipartite graph $\Gamma_{\mathcal{P}}$ the vertex set of which is $\mathcal{P} \cup W$ and with a vertex w adjacent to a vertex $B \in \mathcal{P}$ if and only if w is adjacent in Γ to the vertices in B . Since each $w \in W$ is adjacent to precisely two blocks in \mathcal{P} and each block in \mathcal{P} is adjacent to three vertices in W , this graph is biregular of valence $\{2, 3\}$. Observe also that the action

of G on $V(\Gamma)$ induces a faithful locally arc-transitive action of G on $V(\Gamma_{\mathcal{P}})$ as a group of automorphisms of Λ .

We shall now prove that $\Gamma_{\mathcal{P}}$ contains no cycles of length 4. Indeed, if $B, C \in \mathcal{P}$ and $w, v \in W$ are such that $BwCv$ is a cycle of length 4 in $\Gamma_{\mathcal{P}}$, then w and v are both adjacent to all four vertices of $B \cup C$ in Γ . But then w and v share the same neighbourhood, which contradicts our assumption that no two vertices of degree 4 have that property. This shows that $\Gamma_{\mathcal{P}}$ contains no cycles of length 4.

It is now clear that if one suppresses all the vertices of $\Gamma_{\mathcal{P}}$ of valence 2 (that is, deletes every vertex v of degree 2 and replaces the 2-path having v as a middle vertex by a single edge), a cubic G -arc-transitive graph Λ is obtained. Moreover, it follows easily from the definition of the operator \mathbb{D}_2 and the way we obtained $\Gamma_{\mathcal{P}}$ that $\Gamma \cong \mathbb{D}_2\Lambda$. \square

4. THE LIST OF GRAPHS

As described in Section 1, one can use Theorem 1 to compile a complete list of graphs of valence $\{3, 4\}$ and bounded order admitting a locally arc-transitive group of automorphisms with trivial edge kernel. We have carried out this computation for graph of order at most 350 and thus obtained 220 pairwise non-isomorphic graphs, which are accessible in magma code at [23].

In Table 2 below, some graph-theoretical parameters for the 42 graphs on up to 100 vertices are computed. Each line in the table corresponds to one of the 42 graphs. The first item in each line is the ID of the graph and has form $[n, i]$, where n is the number of the vertices of the graph and i is an index of that graph within the family of all the graphs on n vertices in the table. (The graph with ID $[n, i]$ is stored at [23] in a magma code under the name LAT34 $[n, i]$.) Next three items in each line correspond to the girth (the length of a shortest cycle), diameter, and worthiness of the graph. Then a few parameters pertaining the full automorphism group A of the graph are given. First, for a pair of adjacent vertices v and u of valency 3 and 4, respectively, the permutation groups $A_v^{\Gamma(v)}$ and $A_u^{\Gamma(u)}$ are computed. Further, the parameters s_v and s_u , corresponding to the largest integers s such that A_v (resp. A_u) is transitive on the s -arcs starting at vertex v (resp. u), are given. The next two items are the orders of the edge stabiliser A_{uv} and the edge kernel $A_{uv}^{[1]}$ in A . Note that, since the full automorphism group A might be larger than any of the groups G arising as quotients of the groups U_i , the edge kernel $A_{uv}^{[1]}$ can be non-trivial. The last item in a line is named “comments” and gives extra information, such as, whether the graph is isomorphic to some well-known graph or a subdivided double of a graph appearing in the census of cubic arc-transitive graphs [7] (here Q_3 stands for the graph of the cube, Pet stands for the Petersen graph, and the symbols F014, F016, F018, F020A, F020B, F024, F026, F028 stand for the cubic graphs described in [7]); further comments are provided below.

Let us mention at this point two interesting connections of the topic of this paper with two, at first glance unrelated fields of mathematics.

The first one is the theory of coset geometries which stems from the ingenious work of Tits [26]. A *coset geometry of rank 2* is simply a triple of groups $(G : G_0, G_1)$ with $G_i \leq G$ for $i \in \{0, 1\}$. To such a coset geometry, one can associate an *incidence geometry of rank 2* with points and lines being the cosets of G_0 and G_1 in G , respectively, and a point G_0a being incident with a line G_1b if and only if $G_0a \cap G_1b \neq \emptyset$. Note that the incidence graph of this geometry is precisely the coset

graph $\text{Cos}(G, G_0, G_1)$. Coset geometries of rank 2 and the corresponding incidence structures have received a considerable attention in the last few years (see [10, 11], for example). The results of this paper can be easily interpreted in the language of that particular field of mathematics.

Let us finally mention an interesting relationship of the graphs in Table 2 with flag-transitive configurations. A *configuration of type* $(v_r; b_k)$ is determined by the set \mathcal{P} of v points and a set \mathcal{B} of b subsets of \mathcal{P} of size k such that every point in \mathcal{P} lies in precisely r sets of \mathcal{B} and such that every pair of points lies in at most one set in \mathcal{B} . A pair (x, B) with $x \in \mathcal{P}$, $B \in \mathcal{B}$ and $x \in B$ is called a *flag*. The configuration is *flag-transitive* if the automorphism group of the configuration acts transitively on the flags. The incidence graph of the configuration (also called the *Levi graph*) is the graph with vertex set $\mathcal{P} \cup \mathcal{B}$ and edges of the form xB where (x, B) is a flag of the configuration. It is easy to see that the incidence graph of a flag-transitive configuration is locally arc-transitive and of girth at least 6; conversely, every locally arc-transitive graph of girth at least 6 is an incidence graph of some configuration (we refer to [18] for general information about configurations and to [20] for relationship between flag-transitive configurations and locally arc-transitive graphs).

In the above sense, those graphs in Table 2 that have girth at least 6 correspond to some flag-transitive configurations of type (v_4, b_3) (with those of girth at least 8 corresponding to triangle-free configurations; see [3]). Configurations of type (v_3, b_3) and (v_4, b_4) , also denoted simply by (v_3) and (v_4) , have received by far the greatest attention (see [2, 4] and [1] for enumeration results). Configurations of type (v_4, b_3) (which correspond to graphs of valence $\{3, 4\}$) have also been studied, but much less is known about them (see [17], for example). The graphs in Table 2 can therefore be viewed as a contribution to the theory of flag-transitive configurations of type (v_4, b_3) .

Here are some additional comments:

(1) The graph with ID [14, 2] is the incidence graph of the point–side incidence structure of a cube; that is, the vertices of the graph [14, 2] can be identified with the 8 points and 6 sides of a cube in such a way that edges correspond to incident pairs point–side. Hence the automorphism group of the graph [14, 2] coincides with the group of symmetries of the cube and is thus isomorphic to $S_4 \times C_2$.

(2) The graph with ID [21, 2] is the incidence graph of the point–line geometry of the affine plane \mathbb{Z}_3^2 ; ie. the 9 vertices of valence 4 correspond to the 9 points in \mathbb{Z}_3^2 , the 12 vertices of valence 3 correspond to the 12 lines in \mathbb{Z}_3^2 and a point is adjacent to a line whenever it lies on the line.

(3) The graphs with ID [28, 2] and [28, 3] are the incidence graphs of one of 574 configurations of type $(12_4; 16_3)$ (see [17]). The configuration corresponding to the graph [28, 3] is the so called *Reye configuration*; see for example [25]. It is interesting that both configurations can be realised in the plane by straight lines only. Let us also mention the following interesting construction of the graph [28, 3]. Let $U = \mathbb{Z}_4^2$, and let $W = \{(X, Y) : X \subseteq \mathbb{Z}_4, |X| = 2, Y \in \{X, \mathbb{Z}_4 \setminus X\}\}$. The graph [28, 3] can then be viewed as the graph with vertex set $U \cup W$ and with $(x, y) \in U$ being adjacent to $(X, Y) \in W$ if and only if $x \in X$ and $y \in Y$.

Table 2: The list of connected graphs of valence $\{3, 4\}$ on at most 100 vertices admitting a locally arc-transitive group with a trivial edge kernel

ID	girth	diameter	worthy	$(A_v^{\Gamma(v)}, A_u^{\Gamma(u)})$	(s_v, s_u)	$ A_{uv} $	$ A_{uv}^{[1]} $	comments
[7, 1]	4	2	no	(S_3, S_4)	$(3, 3)$	$2^2 \cdot 3$	1	$K_{3,4}$
[14, 1]	4	4	no	(S_3, D_4)	$(1, 2)$	2^4	2^2	$\mathbb{D}_2(K_4)$
[14, 2]	4	4	yes	(S_3, D_4)	$(1, 2)$	2	1	see (1)
[21, 1]	4	4	no	(S_3, D_4)	$(1, 2)$	2^7	2^3	$\mathbb{D}_2(K_{3,3})$
[21, 2]	6	4	yes	(S_3, S_4)	$(3, 4)$	$2^2 \cdot 3$	1	see (2)
[28, 1]	4	6	no	(S_3, D_4)	$(1, 2)$	2^8	2^6	$\mathbb{D}_2(Q_3)$
[28, 2]	6	4	yes	(S_3, D_4)	$(1, 2)$	2^2	1	see (3)
[28, 3]	6	4	yes	(S_3, S_4)	$(3, 3)$	$2^2 \cdot 3$	1	Reye; see (3)
[35, 1]	4	6	no	(S_3, D_4)	$(1, 2)$	2^{11}	2^9	$\mathbb{D}_2(\text{Pet})$
[35, 2]	6	6	yes	(S_3, S_4)	$(3, 3)$	$2^2 \cdot 3$	1	
[42, 1]	6	4	yes	(S_3, D_4)	$(1, 2)$	2^4	2^2	
[42, 2]	6	4	yes	(S_3, D_4)	$(1, 2)$	2^4	2^2	
[49, 1]	4	6	no	(S_3, D_4)	$(1, 2)$	2^{16}	2^{14}	$\mathbb{D}_2(F[14, 1])$
[49, 2]	6	6	yes	(S_3, D_4)	$(1, 2)$	2	1	
[49, 3]	6	5	yes	(S_3, D_4)	$(1, 2)$	2^2	1	
[56, 1]	4	8	no	(S_3, D_4)	$(1, 2)$	2^{16}	2^{14}	$\mathbb{D}_2(F[16, 1])$
[56, 2]	6	6	yes	(S_3, D_4)	$(1, 2)$	2	1	
[56, 3]	6	6	yes	(S_3, D_4)	$(1, 2)$	2^4	2^2	
[56, 4]	8	6	yes	(S_3, S_4)	$(3, 3)$	$2^2 \cdot 3$	1	
[56, 5]	6	6	yes	(S_3, D_4)	$(1, 2)$	2^2	1	
[63, 1]	4	8	no	(S_3, D_4)	$(1, 2)$	2^{19}	2^{17}	$\mathbb{D}_2(F[18, 1])$
[63, 2]	8	6	yes	(S_3, S_4)	$(3, 3)$	$2^2 \cdot 3$	1	
[63, 3]	8	6	yes	(S_3, D_4)	$(1, 2)$	2	1	
[63, 4]	8	6	yes	(S_3, S_4)	$(3, 4)$	$2^2 \cdot 3$	1	
[70, 1]	4	10	no	(S_3, D_4)	$(1, 2)$	2^{20}	2^{18}	$\mathbb{D}_2(F[20, 1])$
[70, 2]	4	10	no	(S_3, D_4)	$(1, 2)$	2^{21}	2^{19}	$\mathbb{D}_2(F[20, 2])$
[70, 3]	8	6	yes	(S_3, S_4)	$(3, 3)$	$2^2 \cdot 3$	1	
[84, 1]	4	8	no	(S_3, D_4)	$(1, 2)$	2^{24}	2^{22}	$\mathbb{D}_2(F[24, 1])$
[84, 2]	8	6	yes	(S_3, D_4)	$(1, 2)$	2	1	
[84, 3]	8	6	yes	(S_3, \mathbb{Z}_2^2)	$(1, 2)$	2	1	
[84, 4]	6	8	yes	(S_3, D_4)	$(1, 2)$	2^3	2	
[91, 1]	4	10	no	(C_3, D_4)	$(1, 1)$	2^{25}	2^{24}	$\mathbb{D}_2(F[26, 1])$
[91, 2]	8	6	yes	(C_3, C_4)	$(1, 1)$	1	1	
[98, 1]	4	10	no	(S_3, D_4)	$(1, 2)$	2^{29}	2^{27}	$\mathbb{D}_2(F[28, 1])$
[98, 2]	6	6	yes	(S_3, D_4)	$(1, 2)$	2	1	
[98, 3]	6	8	yes	(S_3, D_4)	$(1, 2)$	2	1	
[98, 4]	8	6	yes	(S_3, D_4)	$(1, 2)$	2	1	
[98, 5]	6	6	yes	(S_3, D_4)	$(1, 2)$	2^9	2^7	
[98, 6]	6	6	yes	(S_3, D_4)	$(1, 2)$	2^9	2^7	
[98, 7]	6	6	yes	(S_3, D_4)	$(1, 2)$	2	1	
[98, 8]	8	6	yes	(S_3, D_4)	$(1, 2)$	2^2	1	
[98, 9]	6	6	yes	(S_3, \mathbb{Z}_2^2)	$(1, 2)$	2	1	

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